

Viscosity and thermal conductivity effects on the stability of a compressible plane Couette flow

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Acknowledgements:

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Outline

Modal analysis

- Thermal Conductivity (and it's stratification)
- Viscosity (and it's stratification)
- Combined Viscosity and Thermal Conductivity

Non-Modal analysis

- Effect of stratification of viscosity and conductivity

Variation of Viscosity and Conductivity of Air

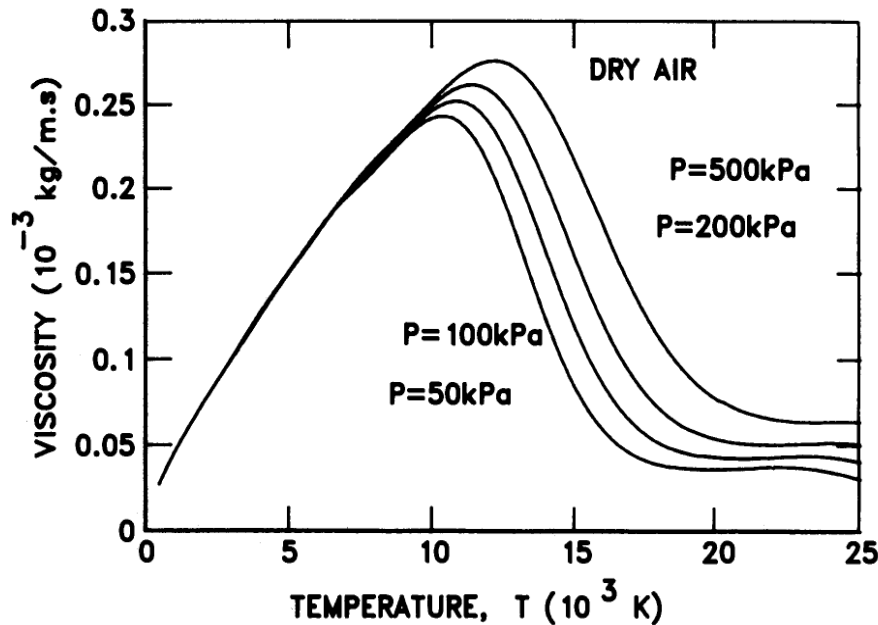


Figure: Variation of viscosity of air with temperature and pressure.

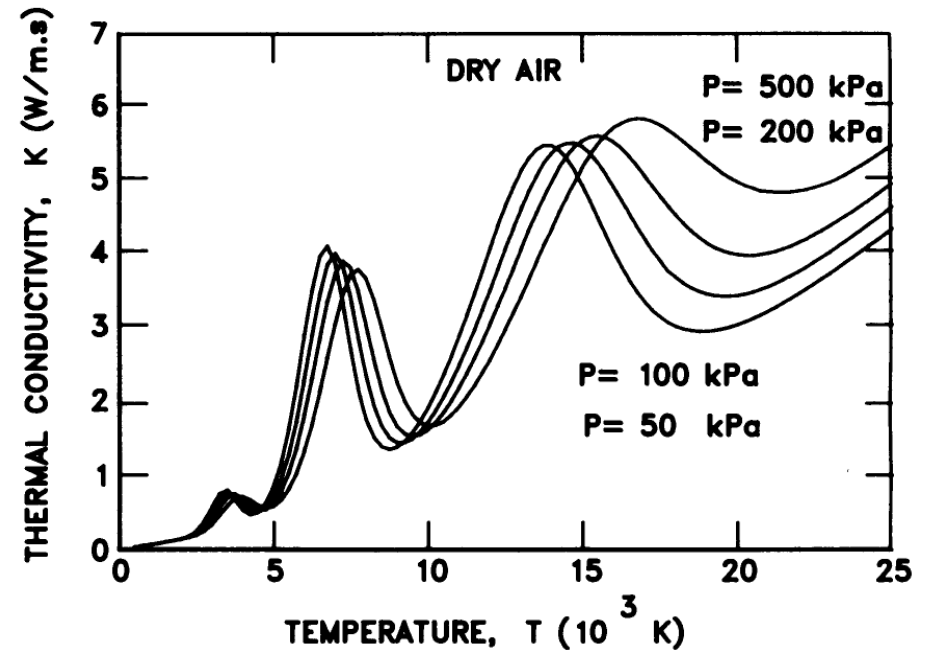


Figure: Variation of thermal conductivity of air with temperature and pressure.

Different variation of viscosity and thermal conductivity with temperature

Modal Analysis : Effect of Thermal conductivity

Base flow governing equations

- **Reference** (Uniform) flow : μ_o , k_o , Pr_{ref}

$$\bar{u} = y$$

$$\bar{T} = 1 + \frac{(\gamma - 1)M^2 Pr}{2} (1 - y)^2$$

- **Stratified** flow : μ_o , $k(T)$, Pr_{ref}

$$\bar{u} = y \quad \bar{\kappa} = \bar{T}^{3/2} \frac{(1 + B)}{(\bar{T} + B)}$$

$$\frac{d\bar{T}}{dy} = -\frac{1}{2} \frac{(\gamma - 1) M^2 Pr y}{\bar{\kappa}}$$

- **Averaged** (Uniform) flow : μ_o , k_{avg} , Pr_{eq}

$$\bar{u} = y \quad \kappa_{avg} = \int_0^1 \kappa dy \quad Pr_{eq} = \frac{Pr}{\kappa_{avg}}$$

$$\bar{T} = 1 + \frac{(\gamma - 1)M^2 Pr}{2} (1 - y)^2$$

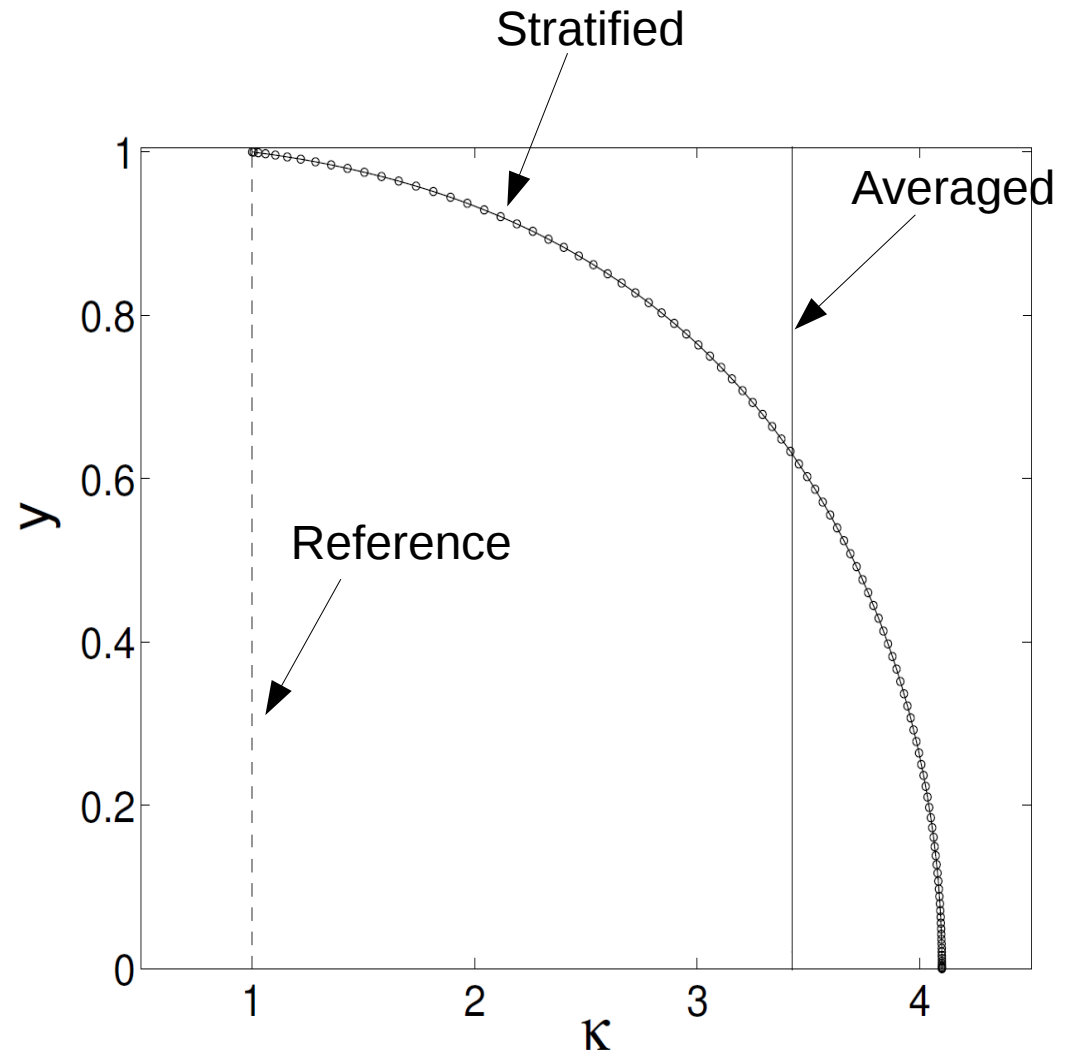


Figure: Variation of thermal conductivity in the wall normal direction for $M=10$, $Pr_{ref}=0.72$.

Base flow profiles

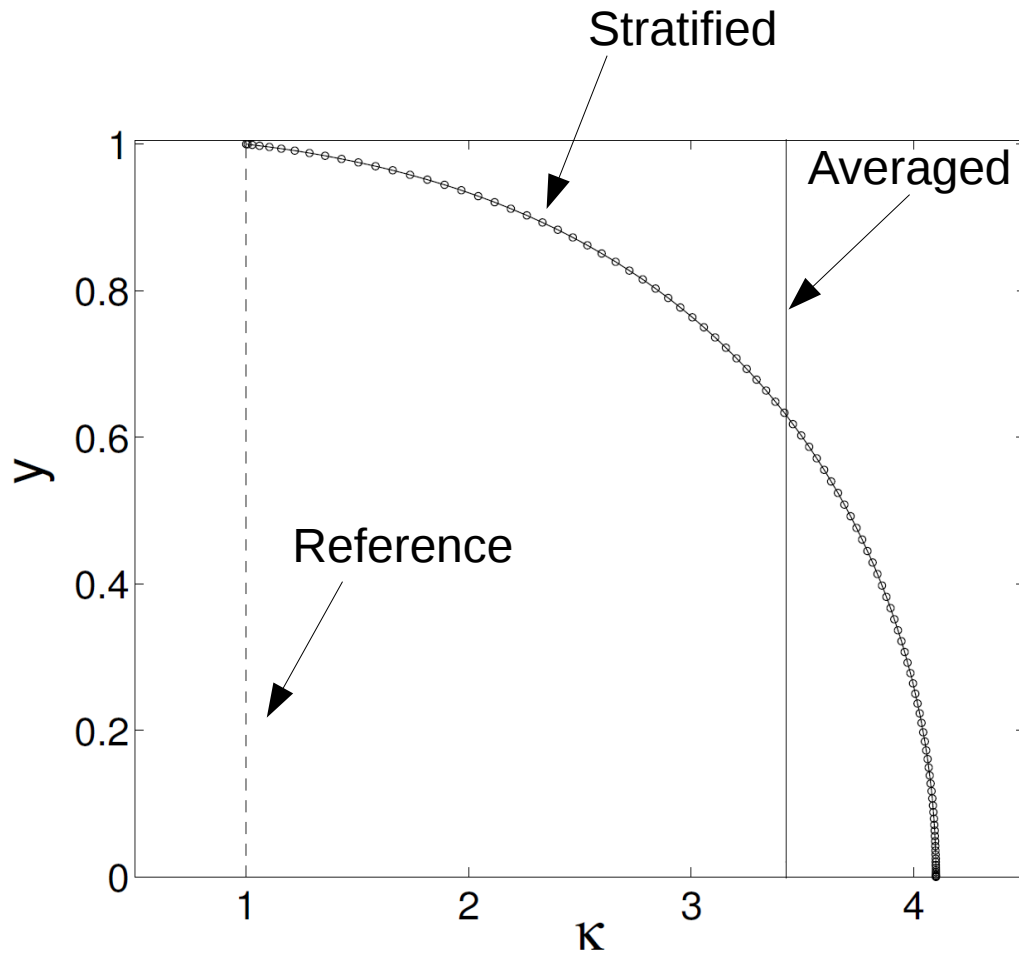


Figure: Variation of thermal conductivity in the wall normal direction for $M=10$, $Pr_{ref}=0.72$.

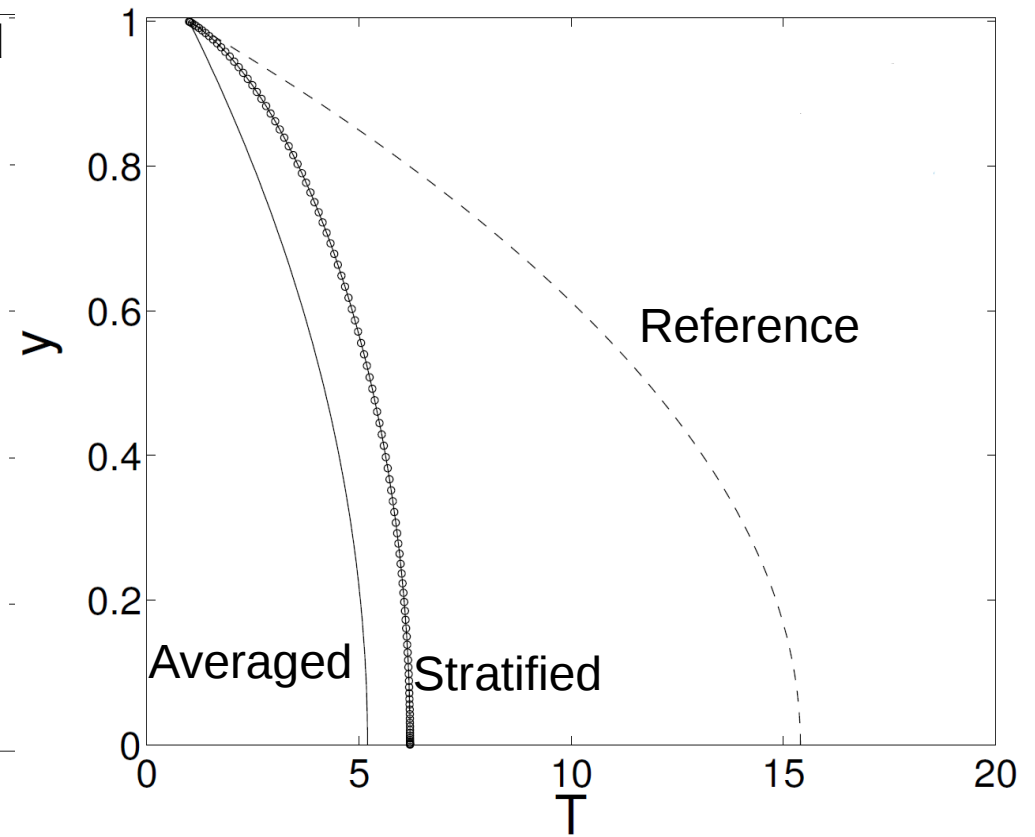
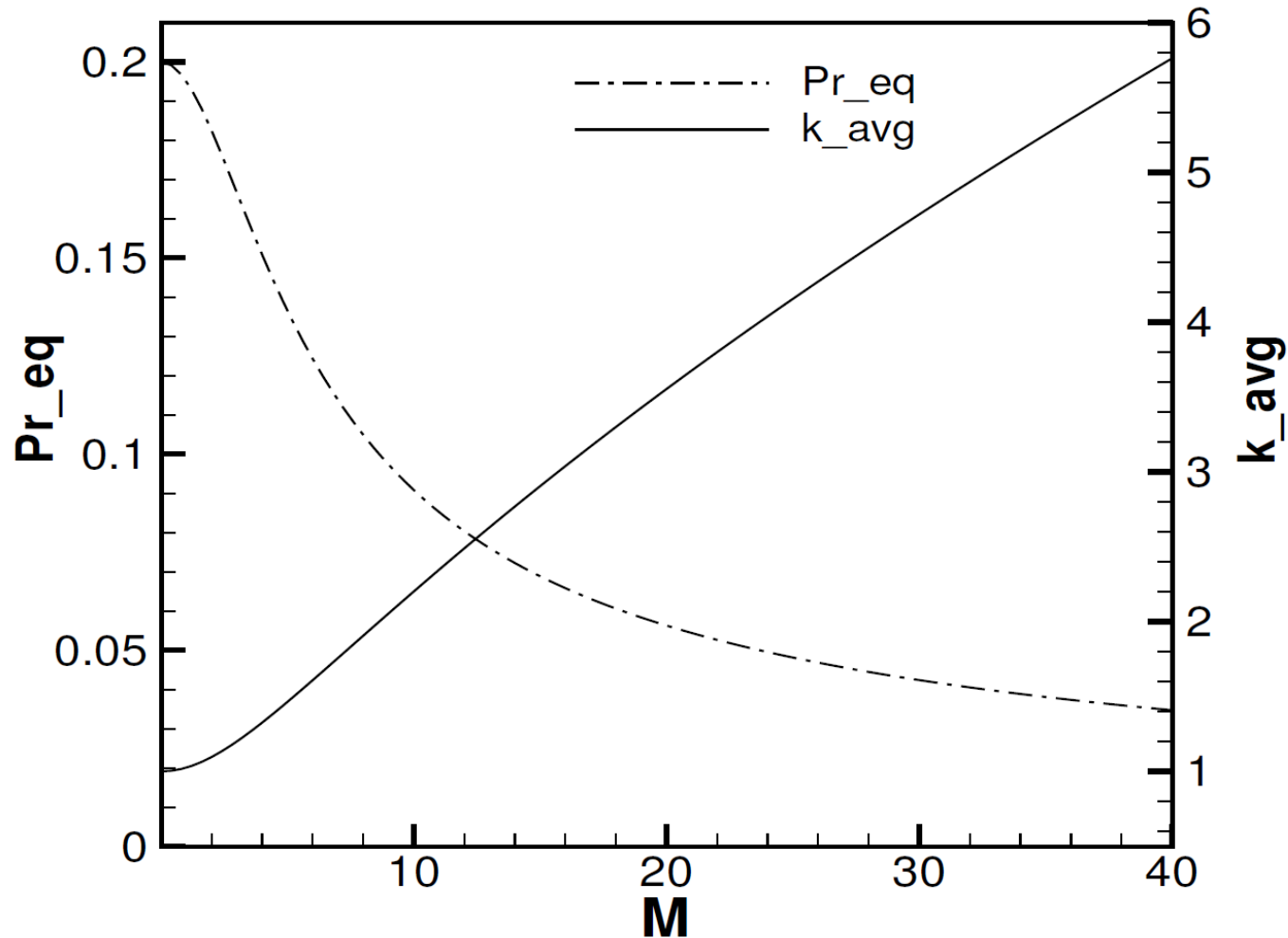


Figure: Variation of temperature in the wall normal direction for $M=10$, $Pr_{ref}=0.72$.

Base flow profiles



$$\bar{\kappa} = \bar{T}^{3/2} \frac{(1 + B)}{(\bar{T} + B)}$$

$$\kappa_{avg} = \int_0^1 \kappa dy$$

$$Pr_{eq} = \frac{Pr}{\kappa_{avg}}$$

Figure: Variation of average thermal conductivity and equivalent Prandtl number of a stratified flow with Mach number with $Pr_{ref} = 0.2$

Stability results: Effect of increase in conductivity

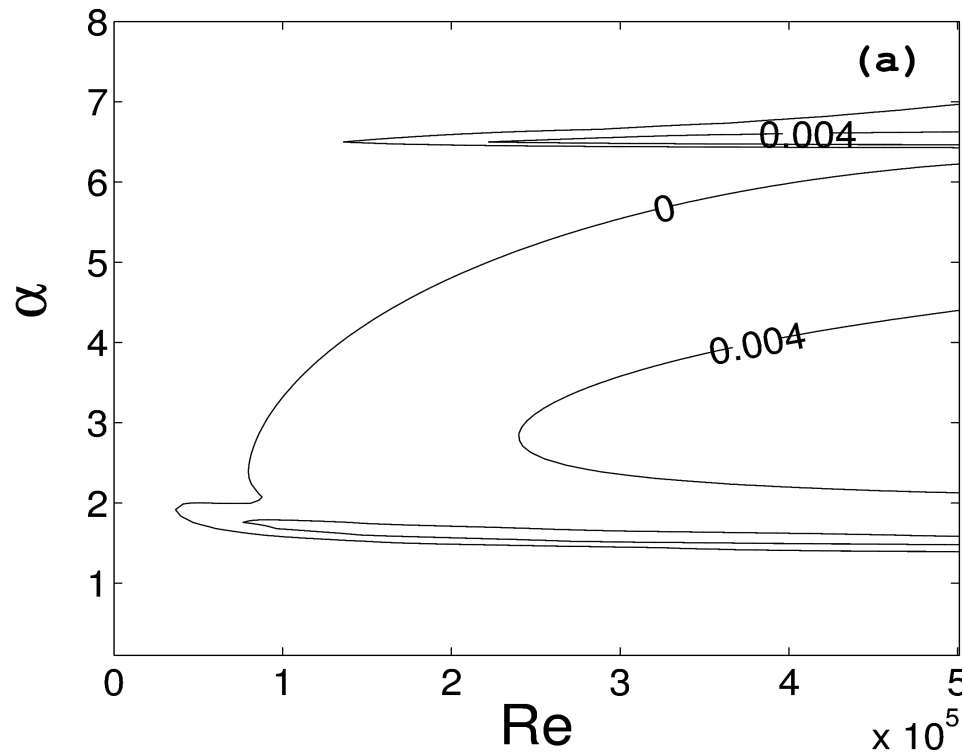


Figure: Neutral curve for **reference** uniform conductivity and viscosity flow at $M=10$, $k_{\text{avg}} = 1$.

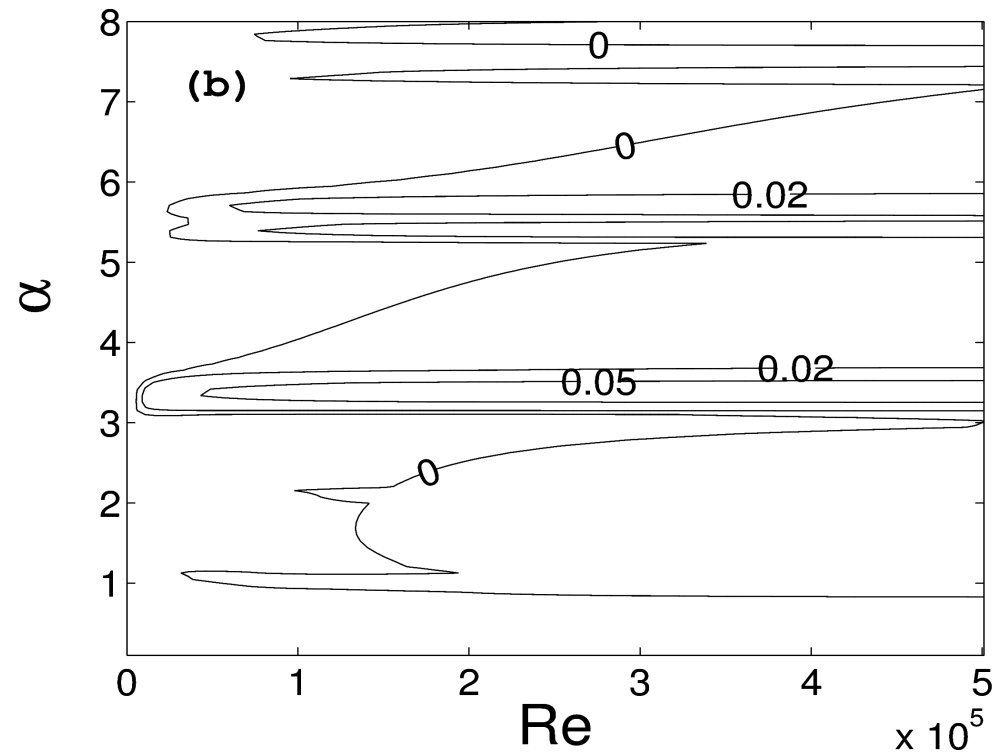


Figure: Neutral curve for **averaged** uniform conductivity and viscosity flow at $M=10$, $k_{\text{avg}} = 2.38$.

Increase in thermal conductivity *destabilizes* the flow

Stability results: Effect of stratification of conductivity

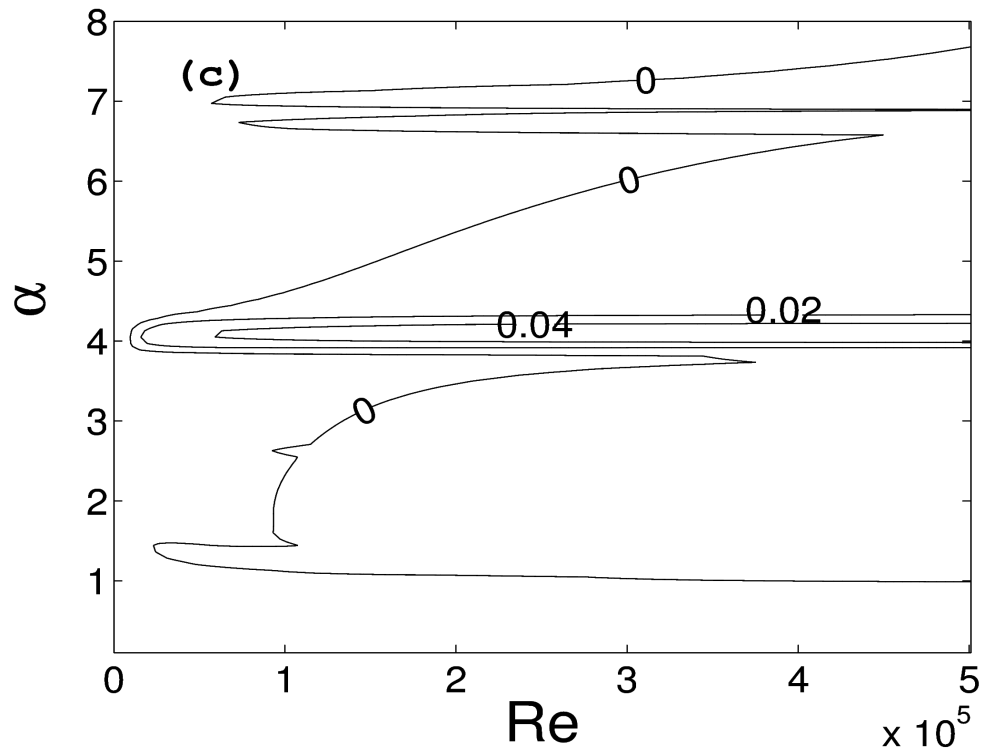


Figure: Neutral curve for **stratified** conductivity and viscosity flow at $M=10$, with $k_{avg} = 2.38$.

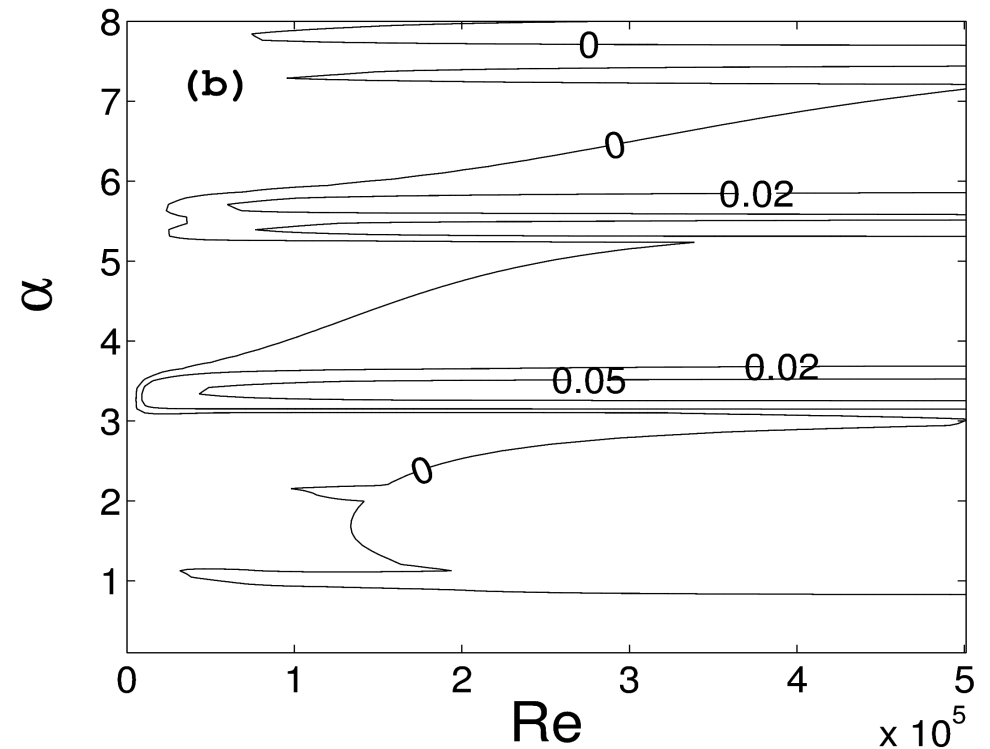


Figure: Neutral curve for **averaged** uniform conductivity and viscosity flow at $M=10$, $k_{avg} = 2.38$.

Stratification in thermal conductivity has *minimal* effect on flow stability

Stability results: Critical Reynolds numbers

M	Re_{cr}			α_{cr}		
	Uniform	Average	Stratified	Uniform	Average	Stratified
3	40555	28051($Pr = 0.46$)	27426	2.58	2.41	2.52
5	19527	13290($Pr = 0.34$)	12252	2.18	1.79	1.97
10	36437	5537($Pr = 0.21$)	8860	1.93	3.41	4.03
15	68620	7173($Pr = 0.16$)	10510	1.86	2.65	3.25



Significant
decrease



Minimal effect,
though mixed



Destabilizing at low M ,
and stabilizing at high M

$Pr_{ref} = 0.5$

A stratified conductivity flow very closely behaves like a uniform flow having the same average value of conductivity in terms of the overall stability. The effect of conductivity stratification is not very significant.

Modal Analysis : Effect of Viscosity

Base flow governing equations

- **Reference** (Uniform) flow : μ_o , k_o , Pr_{ref}

$$\bar{u} = y$$

$$\bar{T} = 1 + \frac{(\gamma - 1)M^2 Pr}{2} (1 - y)^2$$

- **Stratified** flow : $\mu(T)$, k_o , Pr_{ref}

$$\frac{d\bar{u}}{dy} = \frac{\tau}{\mu(\bar{T})} \quad \bar{\mu} = \bar{T}^{3/2} \frac{(1 + C)}{(\bar{T} + C)}$$

$$\frac{d}{dy} \left(\frac{d\bar{T}}{dy} \right) + \frac{1}{2} (\gamma - 1) M^2 Pr \mu \left(\frac{d\bar{u}}{dy} \right)^2 = 0$$

- **Averaged** (Uniform) flow : μ_{avg} , k_o , Pr_{eq}

$$\bar{u} = y \quad \mu_{avg} = \int_0^1 \mu dy \quad Pr_{eq} = \mu_{avg} Pr$$

$$\bar{T} = 1 + \frac{(\gamma - 1)M^2 Pr}{2} (1 - y)^2$$

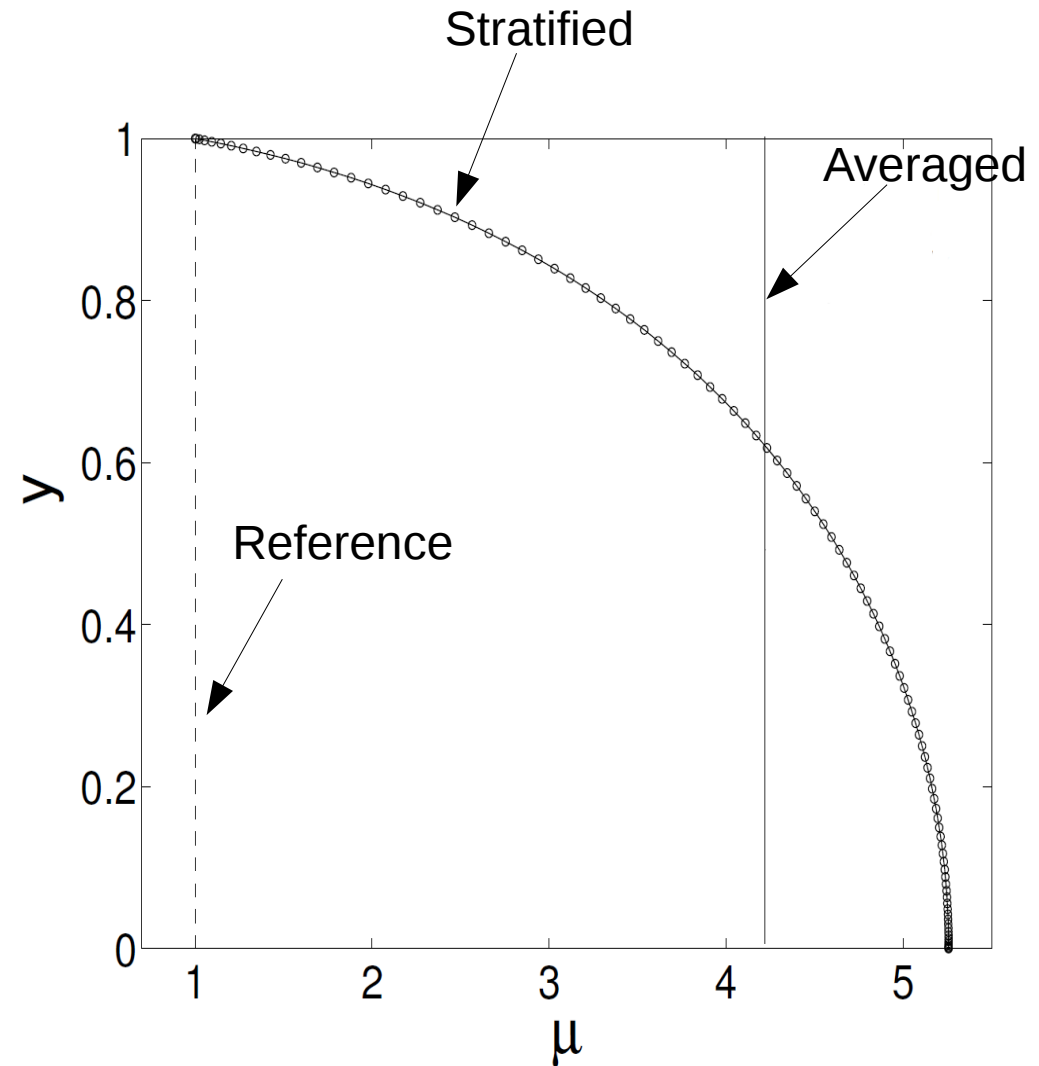


Figure: Variation of viscosity in the wall normal direction for $M=10$, $Pr_{ref}=0.2$.

Base flow profiles

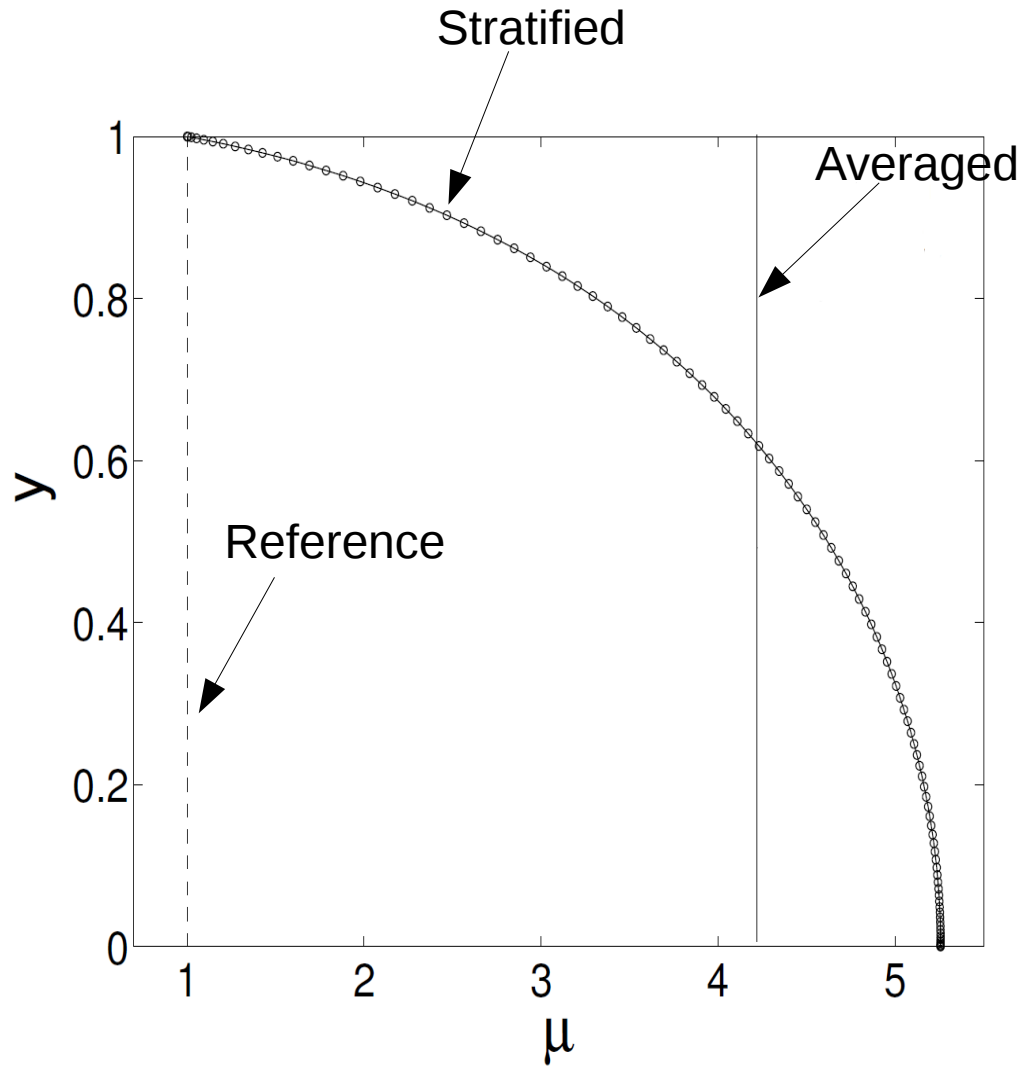


Figure: Variation of viscosity in the wall normal direction for $M=10$, $Pr_{ref}=0.2$.

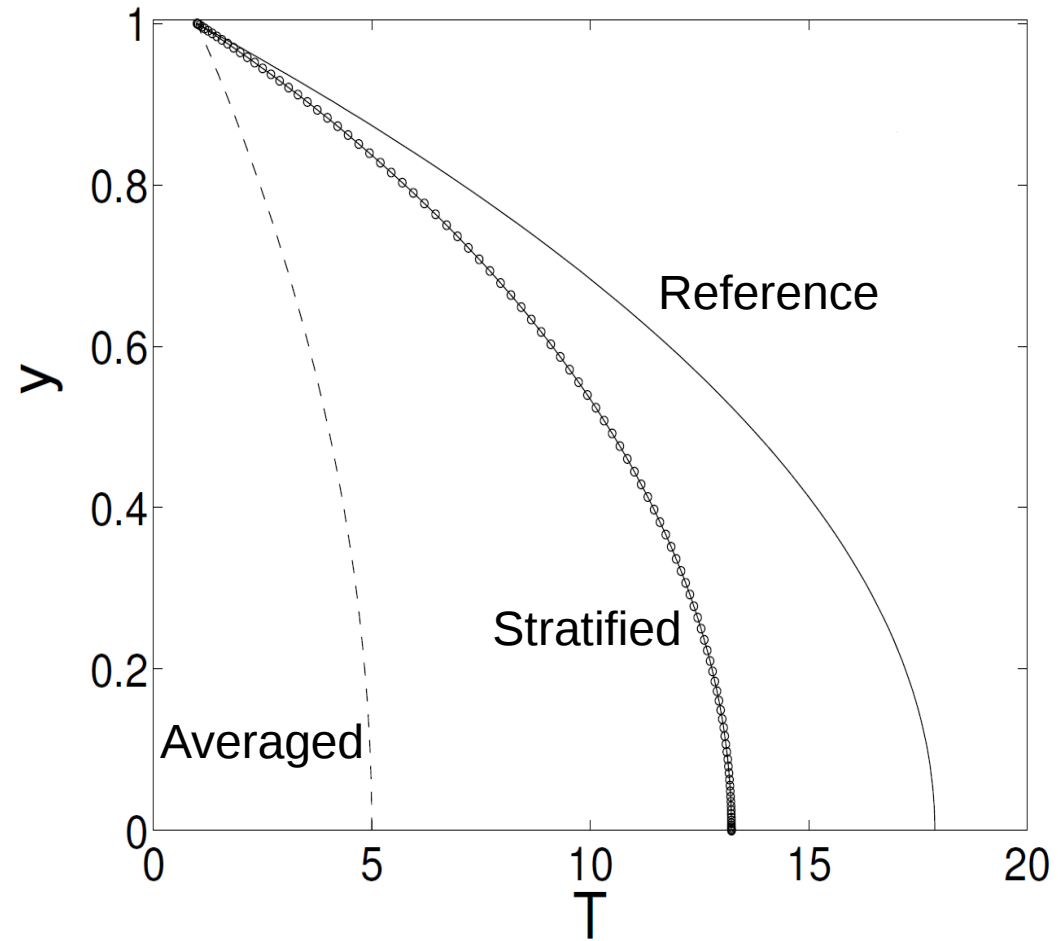
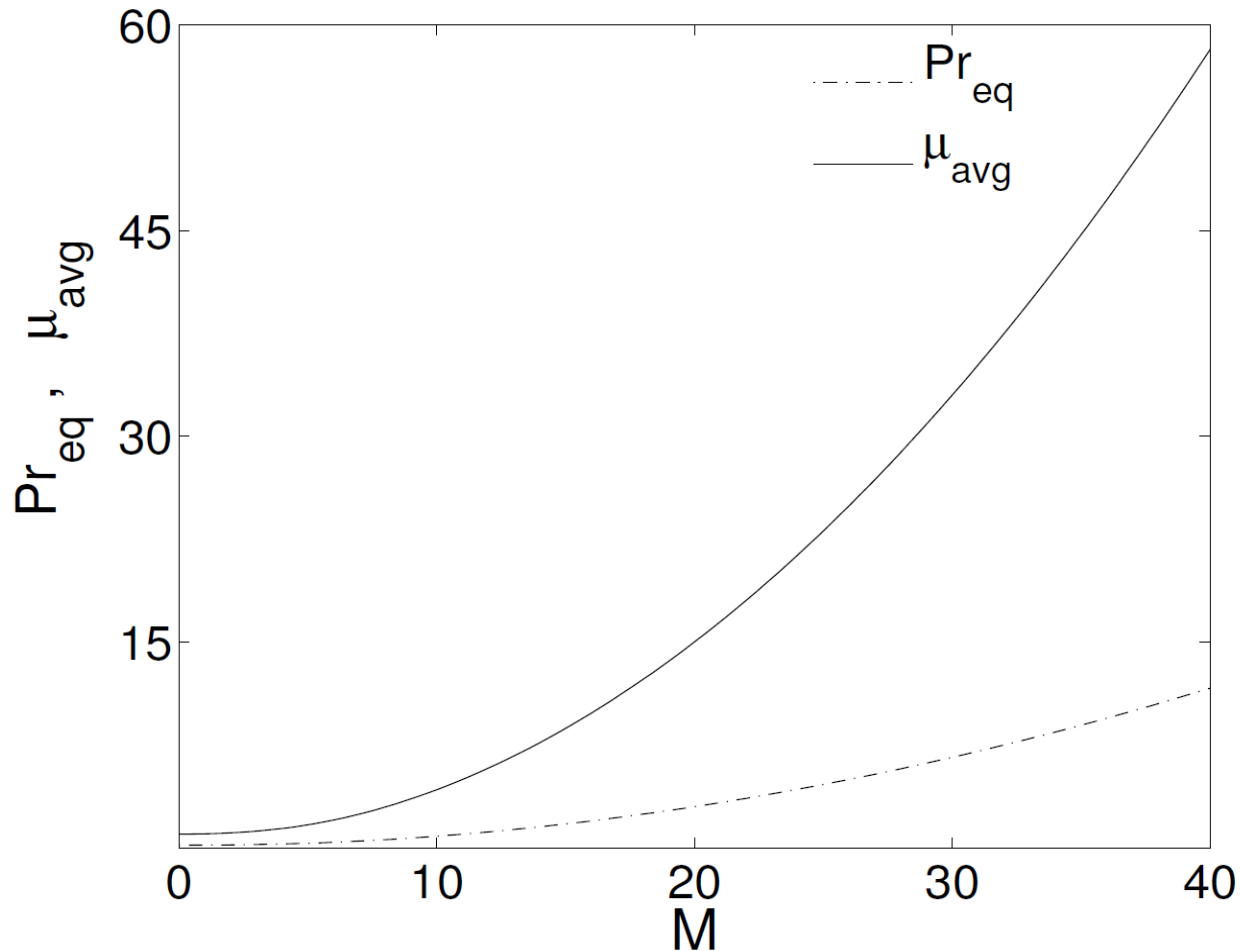


Figure: Variation of temperature in the wall normal direction for $M=10$, $Pr_{ref}=0.2$.

Base flow profiles



$$\bar{\mu} = \bar{T}^{3/2} \frac{(1 + C)}{(\bar{T} + C)}$$

$$\mu_{avg} = \int_0^1 \mu dy$$

$$Pr_{eq} = \mu_{avg} Pr$$

Figure: Variation of average viscosity and equivalent Prandtl number of a stratified flow with Mach number with $Pr_{ref} = 0.2$

Stability results: Effect of increase in viscosity

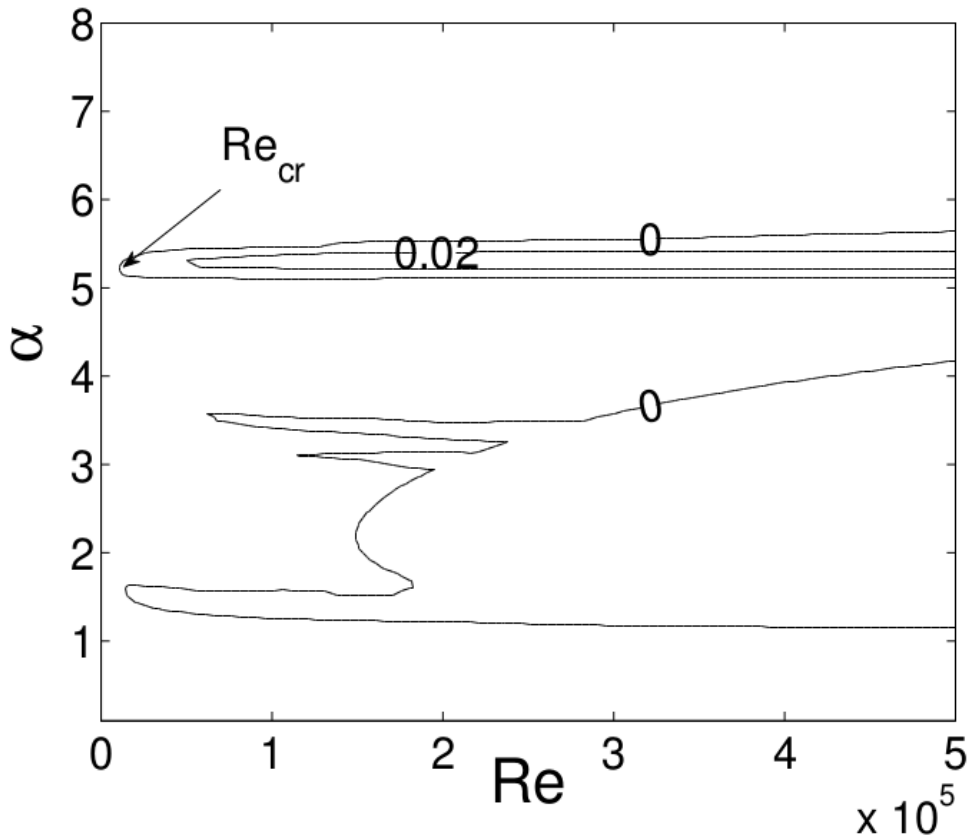


Figure: Neutral curve for **reference** uniform viscosity and conductivity flow at $M=5$, $\mu_{avg} = 1$.

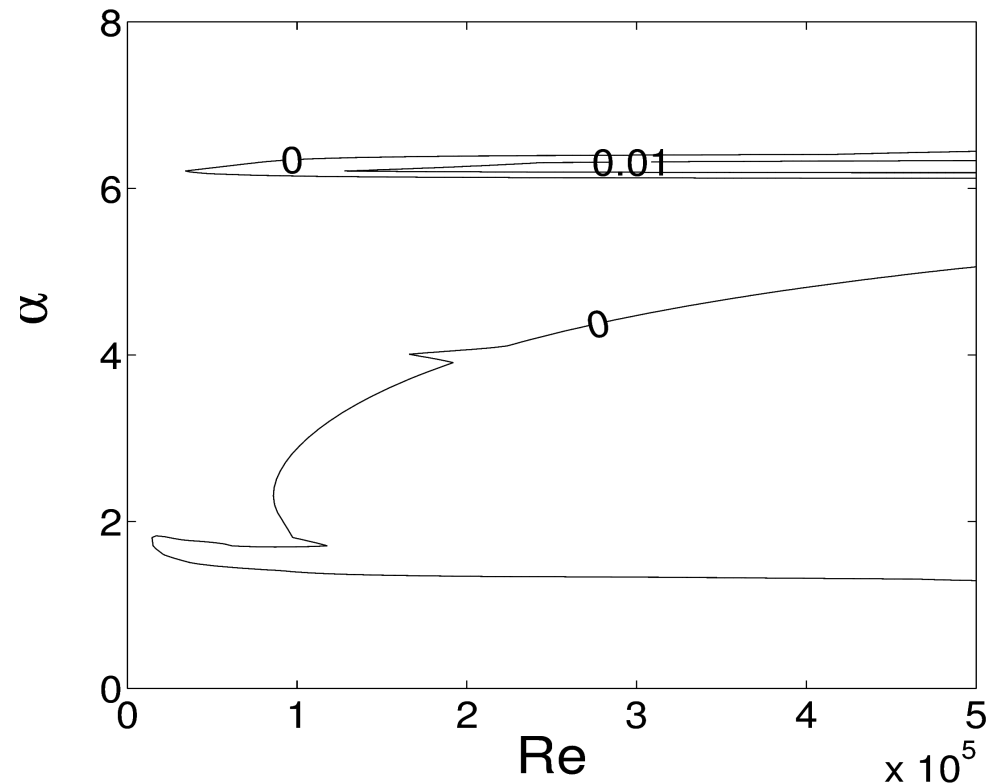


Figure: Neutral curve for **averaged** uniform viscosity and conductivity flow at $M=5$, $\mu_{avg} = 1.67$.

Increase in viscosity *stabilizes* the flow

Stability results: Effect of stratification of viscosity

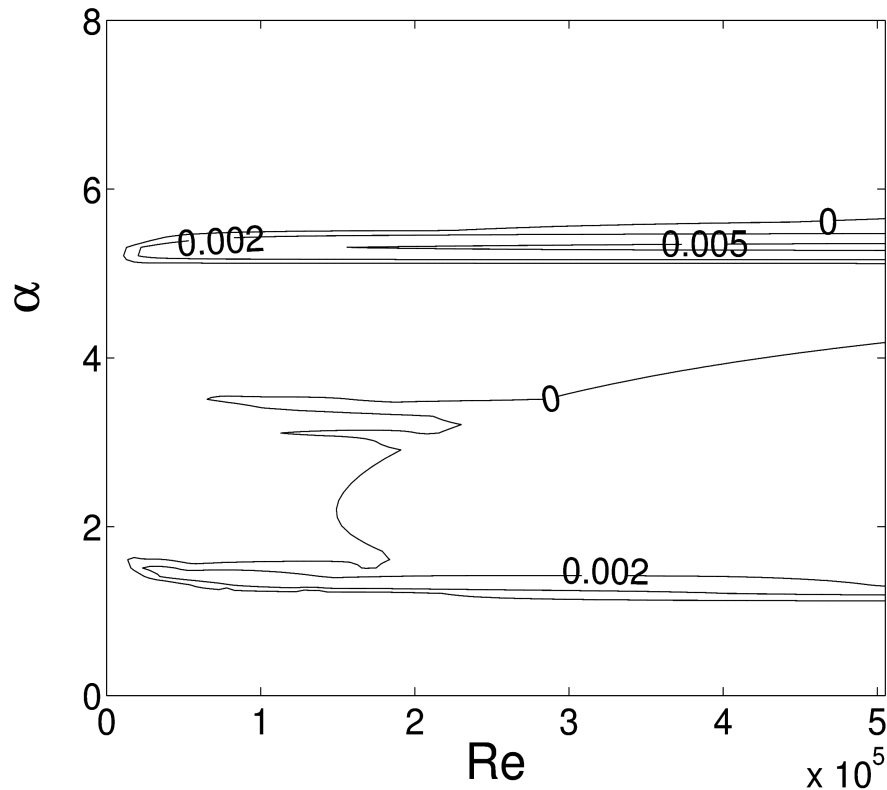


Figure: Neutral curve for uniform conductivity and **stratified** viscosity flow with $M=5$, $\mu_{avg} = 1.67$.

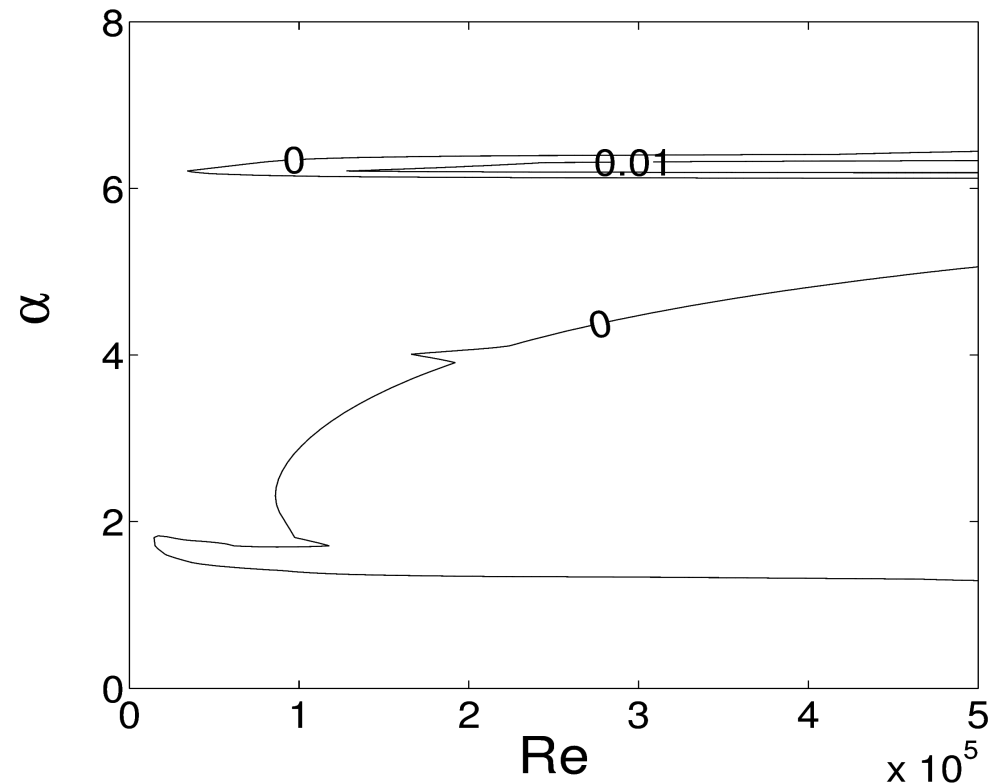
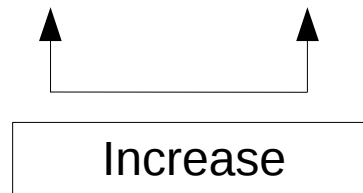
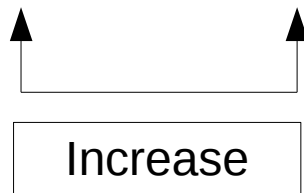


Figure: Neutral curve for **averaged** uniform viscosity and conductivity flow at $M=5$, $\mu_{avg} = 1.67$.

Stratification in viscosity further *stabilizes* the flow

Stability results: Critical Reynolds numbers

M	Re_{cr}			α_{cr}		
	Uniform	Average	Stratified	Uniform	Average	Stratified
3	9704	12148($Pr = 0.2436$)	20784	2.43	2.45	2.26
5	10442	13267($Pr = 0.3352$)	25484	5.22	1.79	1.92
10	5232	41876($Pr = 0.8465$)	187901	3.35	2.05	2.21



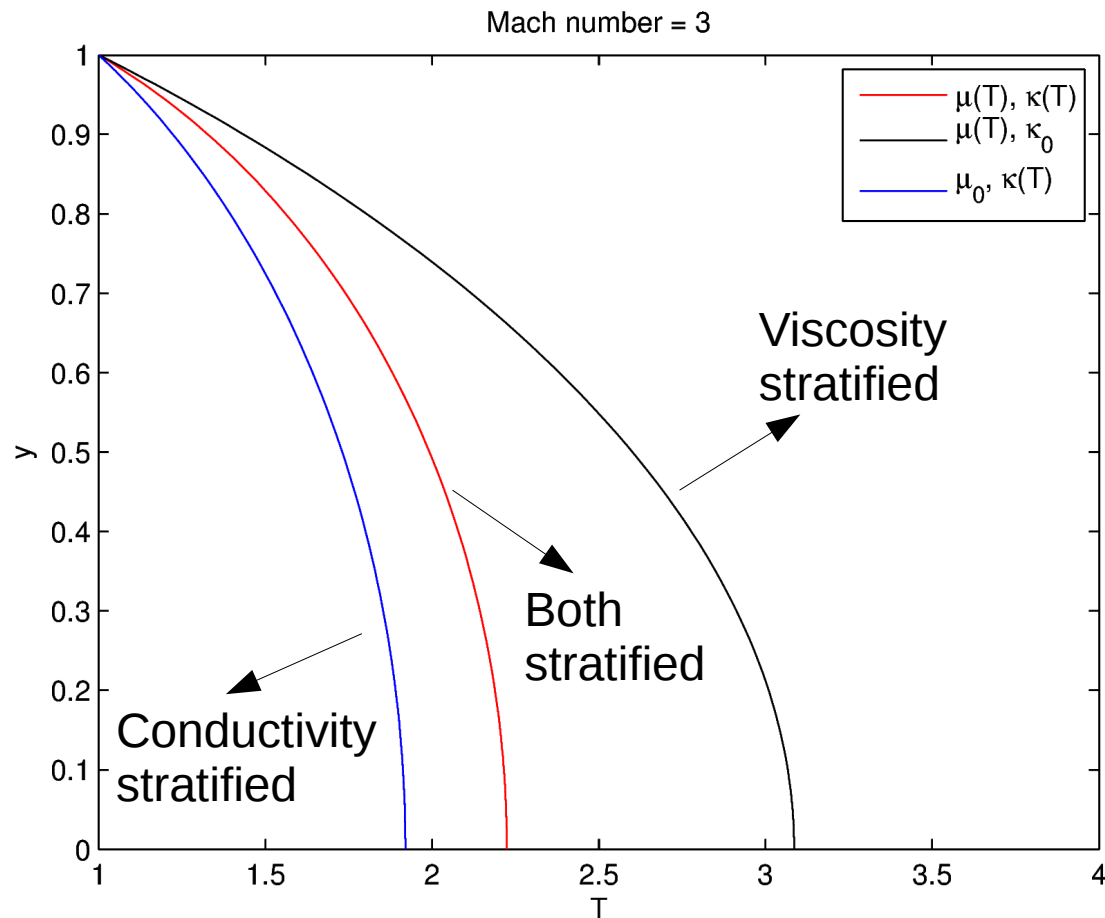
→ Stabilizing at all M

$Pr_{ref} = 0.2$

Both, an increase in the value of viscosity as well as viscosity stratification have got significant stabilizing characteristics.

Modal Analysis : Combined viscosity and conductivity

Base flow governing equations



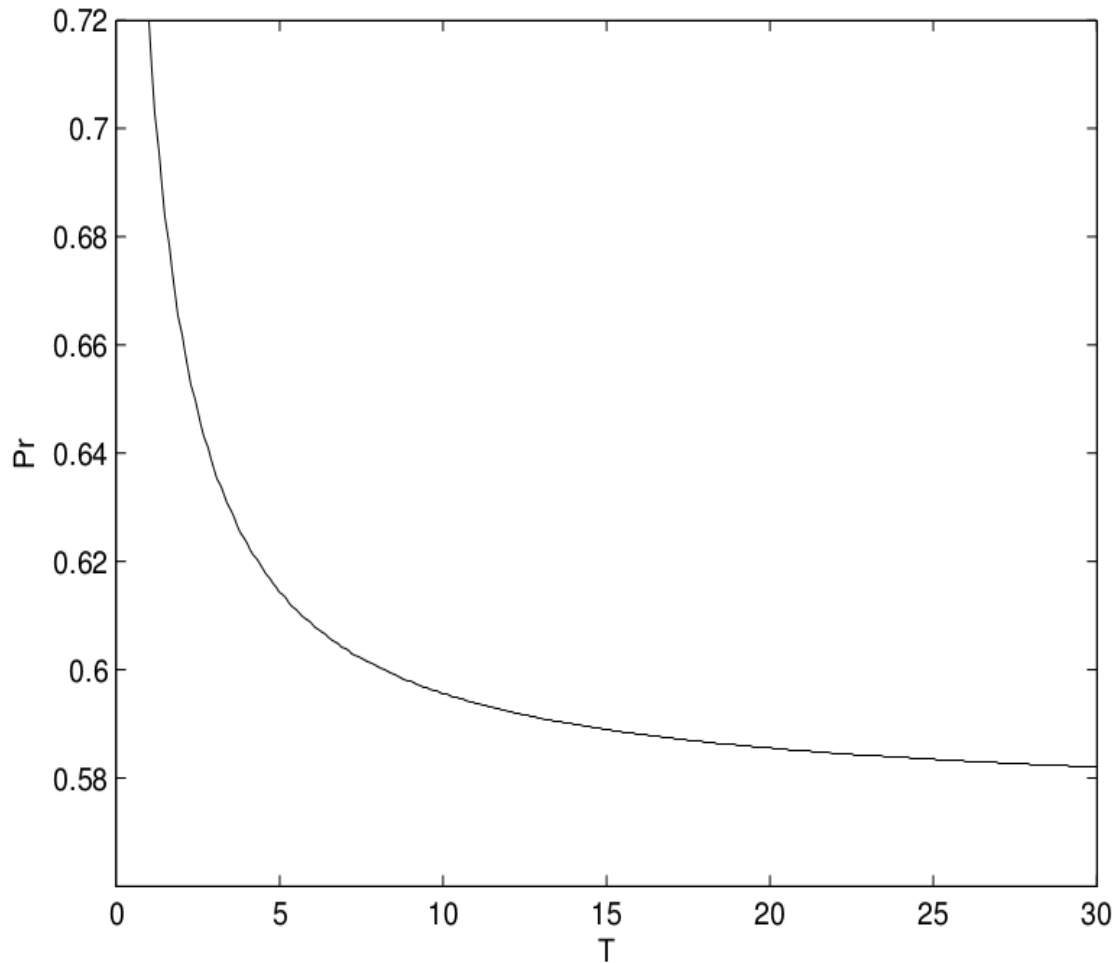
$$\frac{d^2 T}{dy^2} + \frac{1}{\kappa} \frac{d\kappa}{dy} \frac{dT}{dy} + \frac{(\gamma - 1) M^2 Pr \tau^2}{2\mu\kappa} = 0$$

$$\bar{\mu} = \bar{T}^{3/2} \frac{(1 + C)}{(\bar{T} + C)}$$

$$\bar{\kappa} = \bar{T}^{3/2} \frac{(1 + B)}{(\bar{T} + B)}$$

Figure: Variation of temperature along the wall normal direction for $M=3$, $Pr_{ref}=0.72$ for different base flow cases.

Base flow governing equations



$$\bar{\mu} = \bar{T}^{3/2} \frac{(1 + C)}{(\bar{T} + C)}$$

$$\bar{\kappa} = \bar{T}^{3/2} \frac{(1 + B)}{(\bar{T} + B)}$$

At high temperatures,

$$Pr \approx 0.72 \times \frac{(1 + C)}{(1 + B)} = 0.575$$

Figure: Variation of Prandtl number with temperature for the current model problem $Pr_{ref}=0.72$.

Stability results: Effect of simultaneous increase in viscosity and conductivity – Prandtl number change

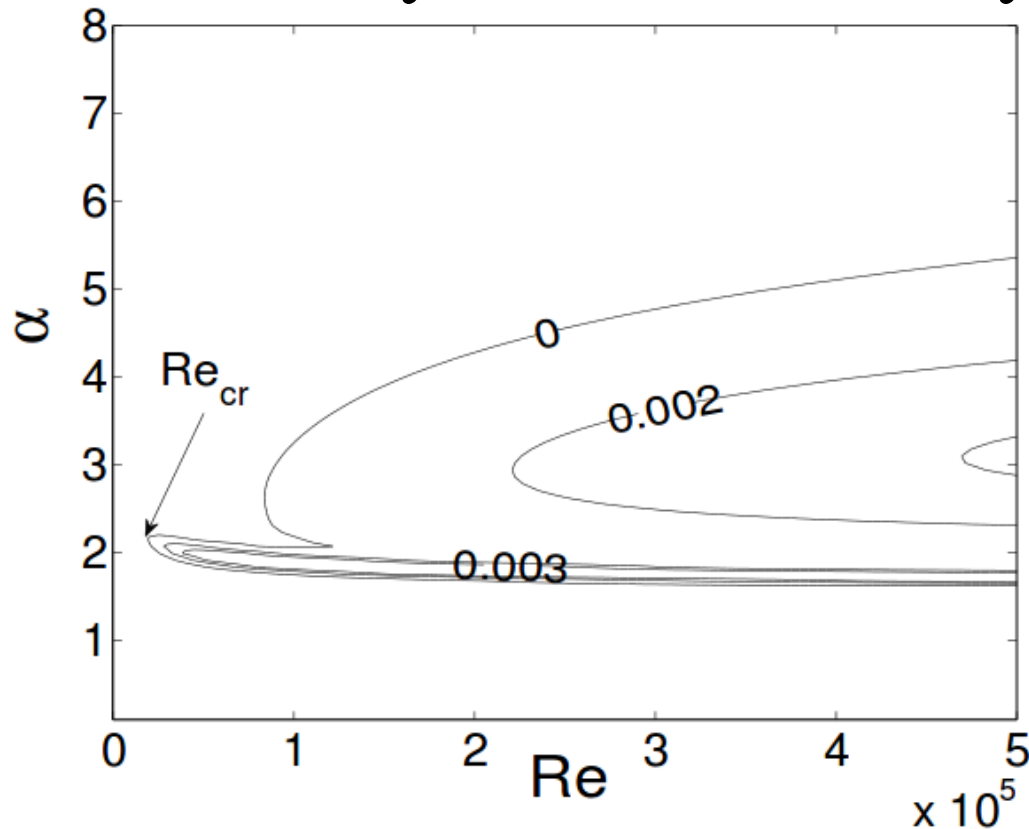


Figure: Neutral curve for **reference** uniform viscosity and conductivity flow at $M=5$, $\mu_{avg} = 1$, $k_{avg} = 1$, $Pr_{ref} = 0.72$.

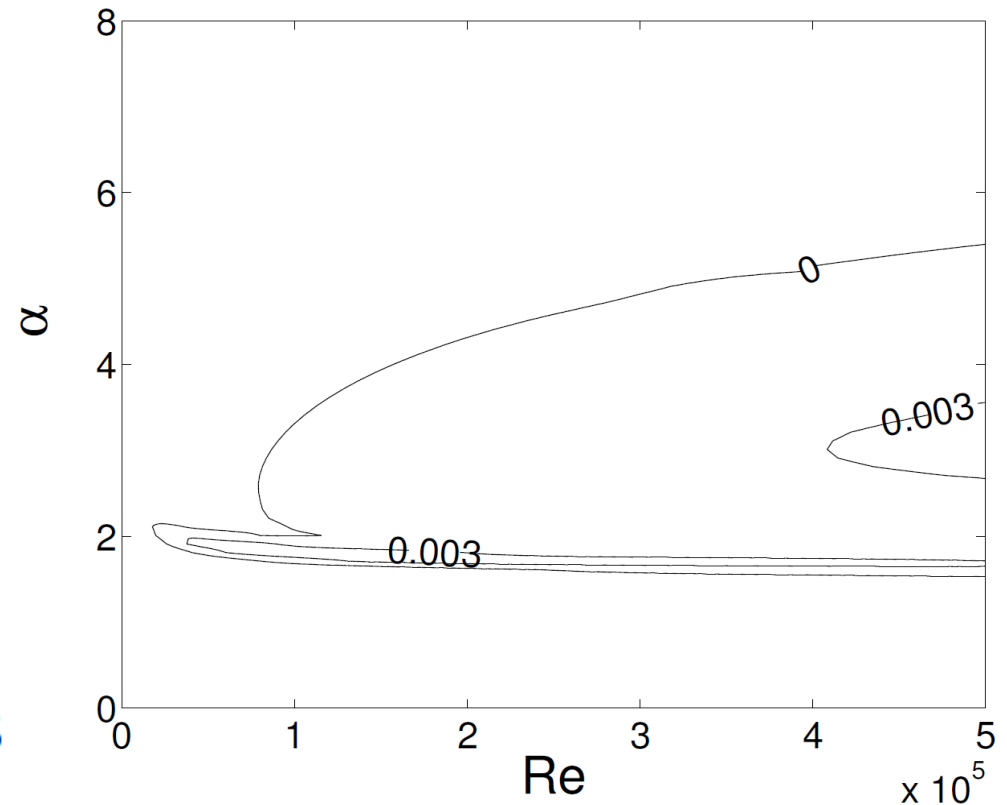


Figure: Neutral curve for **averaged** uniform viscosity and conductivity flow at $M=5$, $\mu_{avg} = 1.67$, $k_{avg} = 1.91$, $Pr_{eq} = 0.63$.

No significant effect due to small change in the equivalent Prandtl number

Stability results: Effect of simultaneous stratification of viscosity and conductivity

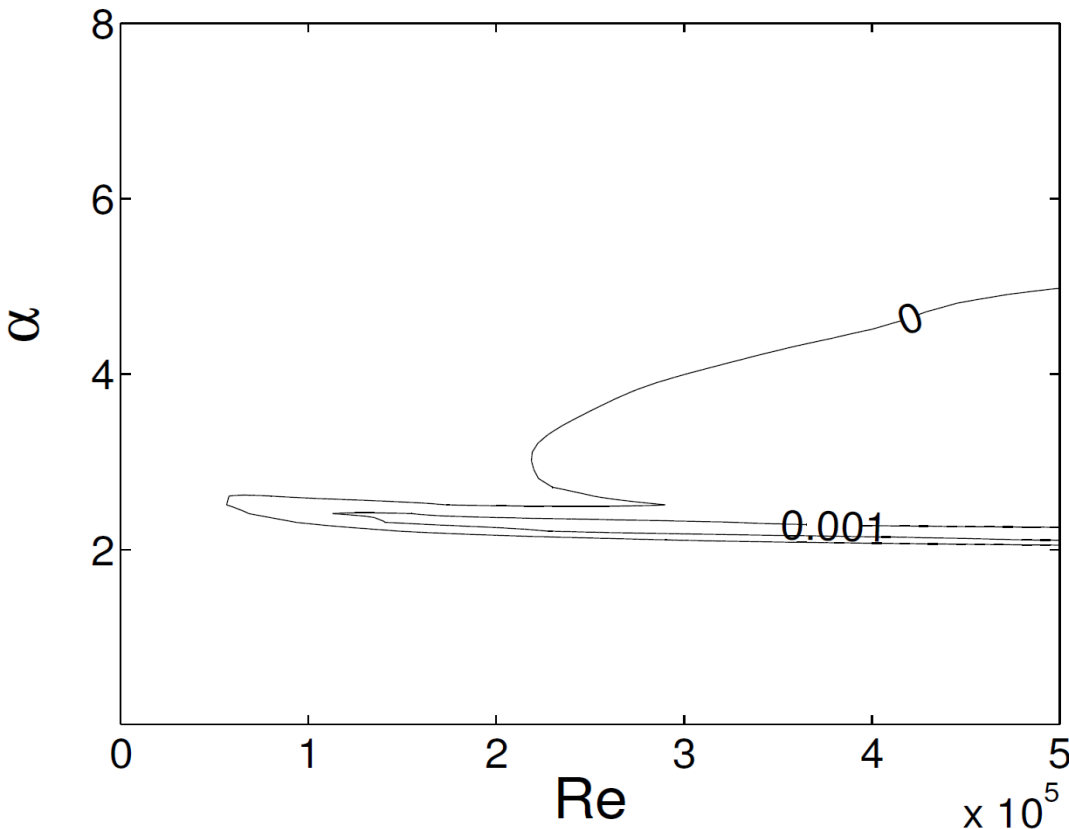


Figure: Neutral curve for **stratified** viscosity and conductivity flow at $M=5$, $M=5$, $\mu_{\text{avg}} = 1.67$, $k_{\text{avg}} = 1.91$.

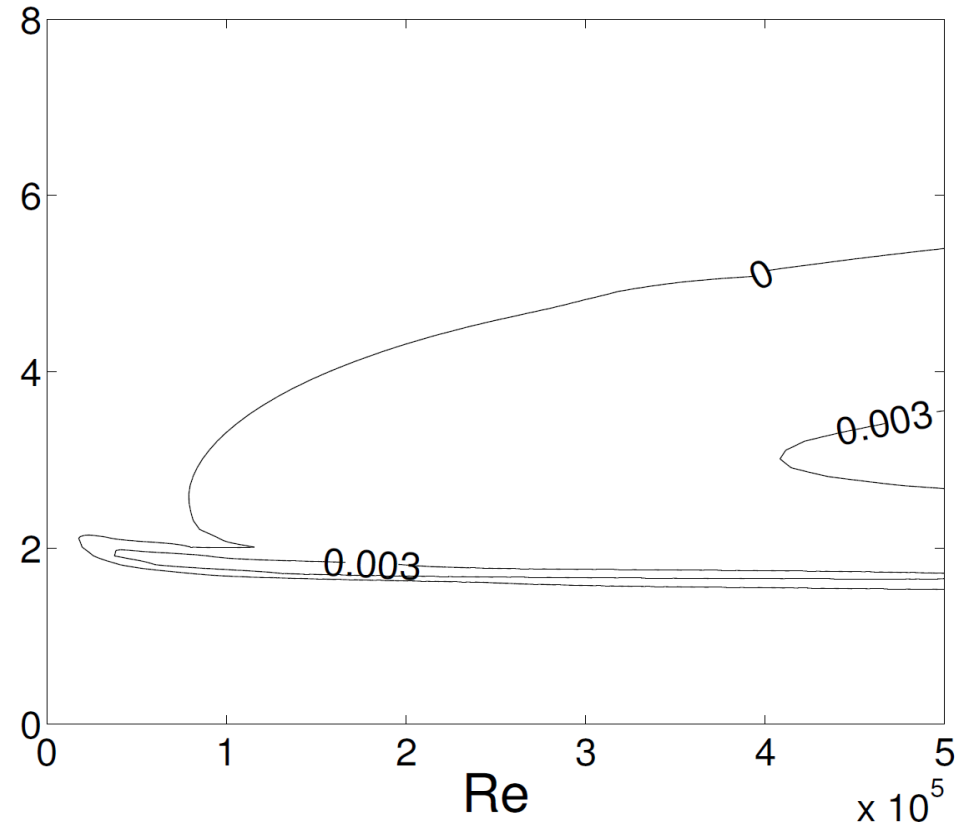


Figure: Neutral curve for **averaged** uniform viscosity and conductivity flow at $M=5$, $\mu_{\text{avg}} = 1.67$, $k_{\text{avg}} = 1.91$.

Stratified flow is more stable than a uniform flow with the same average values of viscosity and conductivity

Stability results: Critical Reynolds numbers

M	Re_{cr}			α_{cr}		
	Uniform	Average	Stratified	Uniform	Average	Stratified
3	40555	37323($Pr = 0.6662$)	111026	2.58	2.55	2.85
5	19527	17497($Pr = 0.6310$)	54115	2.18	2.1	2.57
10	36437	32333($Pr = 0.5961$)	131551	1.93	1.79	2.45

↑ ↑
Small variations

↑ ↑
Large increase

When viscosity and conductivity are stratified, the flow is *more stable* than a flow with the same respective average values.

**Does stratification of viscosity or conductivity
dominate over the other?**

Eigenvalue problem

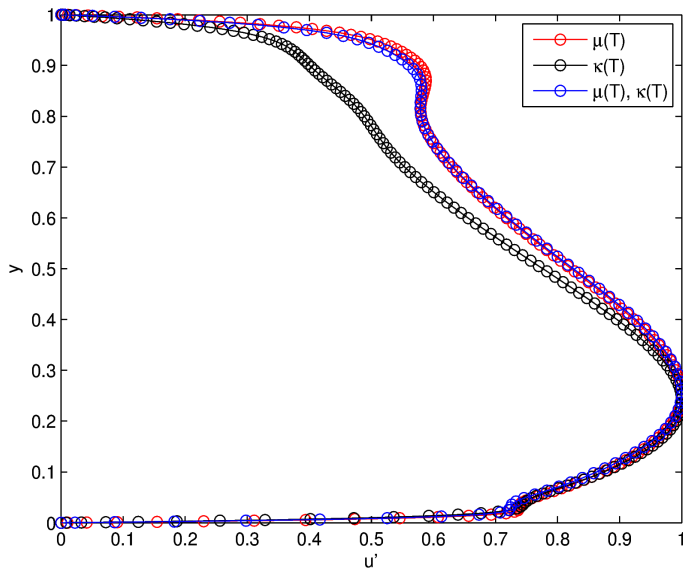
$$\begin{bmatrix} B_{11} & B_{12} & \dots & B_{15} \\ B_{21} & B_{22} & \dots & B_{25} \\ \vdots & & \dots & \vdots \\ \vdots & & \dots & \vdots \\ B_{41} & B_{42} & \dots & B_{45} \\ B_{51} & B_{52} & \dots & B_{55} \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\rho} \\ \hat{T} \end{Bmatrix} = \omega \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \dots & \vdots \\ \vdots & & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \\ \hat{\rho} \\ \hat{T} \end{Bmatrix}$$

Matrix entries:

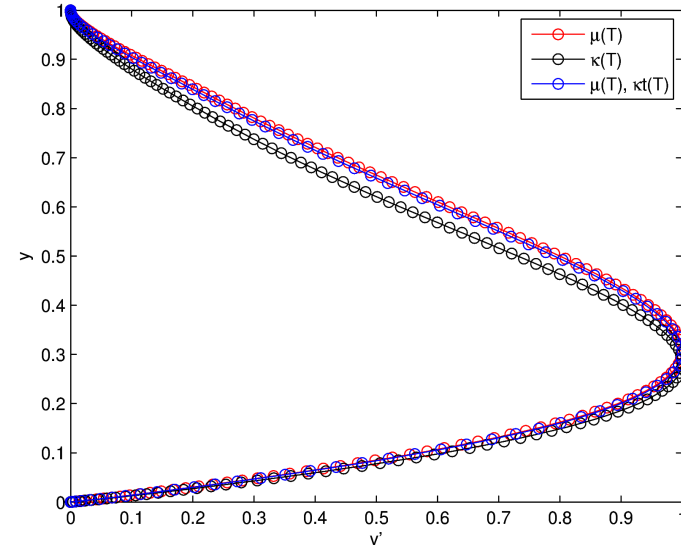
$$\begin{aligned} B_{11} &= \alpha \bar{u} - \iota \frac{[\bar{\mu}(\alpha^2 + \beta^2) + \alpha^2(\bar{\mu} + \bar{\lambda}) - \bar{\mu}_y D - \bar{\mu} D^2]}{\bar{\rho} Re}, \\ B_{12} &= -\iota \bar{u}_y - \alpha \frac{[\bar{\mu}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\ B_{13} &= -\iota \alpha \beta \frac{(\bar{\lambda} + \bar{\mu})}{\bar{\rho} Re}, \\ B_{14} &= \frac{\alpha \bar{T}^2}{\gamma M^2}, \\ B_{15} &= \frac{\alpha}{\gamma M^2} + \iota \frac{[\bar{u}_{yy} \bar{\mu}_T + \bar{u}_y \bar{T}_y \bar{\mu}_{TT} + \bar{u}_y \bar{\mu}_T D]}{\bar{\rho} Re}, \\ B_{21} &= -\alpha \frac{[\bar{\lambda}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\ B_{22} &= \alpha \bar{u} + \iota \frac{[(\bar{\mu}_y + \bar{\lambda}_y) D - \bar{\mu}(\alpha^2 + \beta^2) + (\bar{\mu} + \bar{\lambda}) D^2 + \bar{\mu}_y D + \bar{\mu} D^2]}{\bar{\rho} Re}, \\ B_{23} &= -\beta \frac{[\bar{\mu}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \end{aligned}$$

$$\begin{aligned} B_{24} &= -\iota \frac{(\bar{T}_y + \bar{T} D)}{\bar{\rho} \gamma M^2}, \\ B_{25} &= -\iota \frac{(\bar{\rho}_y + \bar{\rho} D)}{\bar{\rho} \gamma M^2} - \frac{\alpha \bar{u}_y \bar{\mu}_T}{\bar{\rho} Re}, \\ B_{31} &= -\iota \alpha \beta \frac{(\bar{\lambda} + \bar{\mu})}{\bar{\rho} Re}, \\ B_{32} &= -\beta \frac{[\bar{\mu}_y + (\bar{\lambda} + \bar{\mu}) D]}{\bar{\rho} Re}, \\ B_{33} &= \alpha \bar{u} - \iota \frac{[\bar{\mu}(\alpha^2 + \beta^2) + \beta^2(\bar{\mu} + \bar{\lambda}) - \bar{\mu}_y D - \bar{\mu} D^2]}{\bar{\rho} Re}, \\ B_{34} &= \frac{\beta \bar{T}^2}{\gamma M^2}, \\ B_{35} &= \frac{\beta}{\gamma M^2}, \\ B_{41} &= \alpha \bar{\rho}, \\ B_{42} &= -\iota (\bar{\rho}_y + \bar{\rho} D), \\ B_{43} &= \beta \bar{\rho}, \\ B_{44} &= \alpha \bar{u}, \\ B_{45} &= 0, \\ B_{51} &= \alpha (\gamma - 1) \bar{T} + \iota \frac{2(\gamma - 1) M^2 \bar{\mu}_y D}{\bar{\rho} Re}, \\ B_{52} &= -\iota \bar{T}_y - \iota (\gamma - 1) \bar{T} D - \frac{2\alpha (\gamma - 1) M^2 \bar{\mu}_y}{\bar{\rho} Re}, \\ B_{53} &= \beta (\gamma - 1) \bar{T}, \\ B_{54} &= 0, \\ B_{55} &= \alpha \bar{u} + \iota \frac{[\bar{T}_{yy} \bar{k}_T + \bar{T}_y^2 \bar{k}_{TT} + 2\bar{T}_y \bar{k}_T D - (\alpha^2 + \beta^2) \bar{k} + \bar{k} D^2 + (\gamma - 1) M^2 Pr \bar{u}_y^2 \bar{\mu}_T]}{\bar{\rho} Re Pr}. \end{aligned}$$

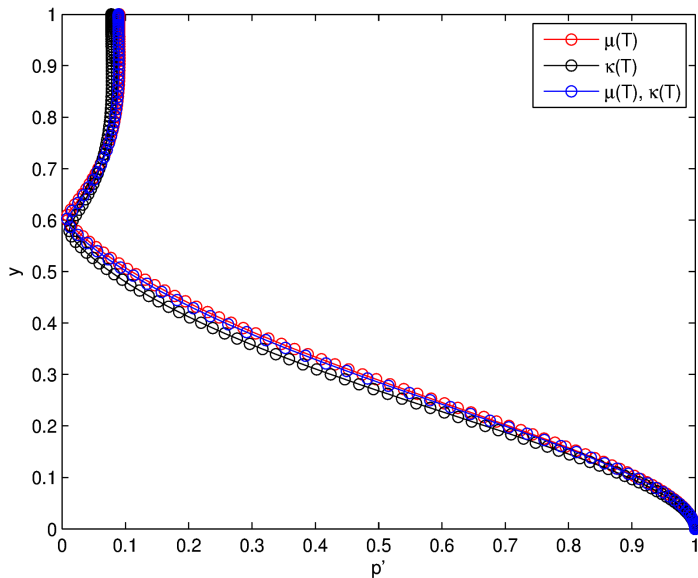
Eigenfunctions for the neutral mode



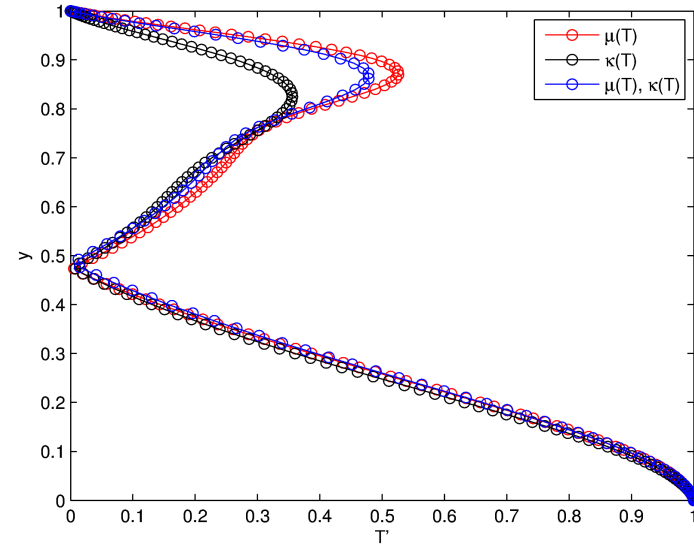
a) u-velocity



b) v-velocity



c) pressure



d) temperature

Figure: Eigenfunctions for the neutral mode at $M=3$, $Pr_{ref} = 0.72$

Modal analysis: Summary

Uniform properties flow :

- Increase in conductivity *destabilizes* the flow
- Increase in viscosity *stabilizes* the flow
- Both these effects can be studied together using the *Prandtl number*.

Stratified properties flow :

- Stratification in conductivity has *minimal effects* on flow stability
- Stratification in viscosity *highly stabilizes* the flow
- When both are stratified, viscosity stratification has a more dominant role in stabilizing the flow.

Non-modal analysis

References:

1. Malik, M., J. Dey, and Meheboob Alam. "*Linear stability, transient energy growth, and the role of viscosity stratification in compressible plane Couette flow.*" *Physical Review E* 77.3 (2008): 036322.
2. Malik, M., Meheboob Alam, and J. Dey. "*Nonmodal energy growth and optimal perturbations in compressible plane Couette flow.*" *Physics of Fluids (1994-present)* 18.3 (2006): 034103.

Transient energy growth

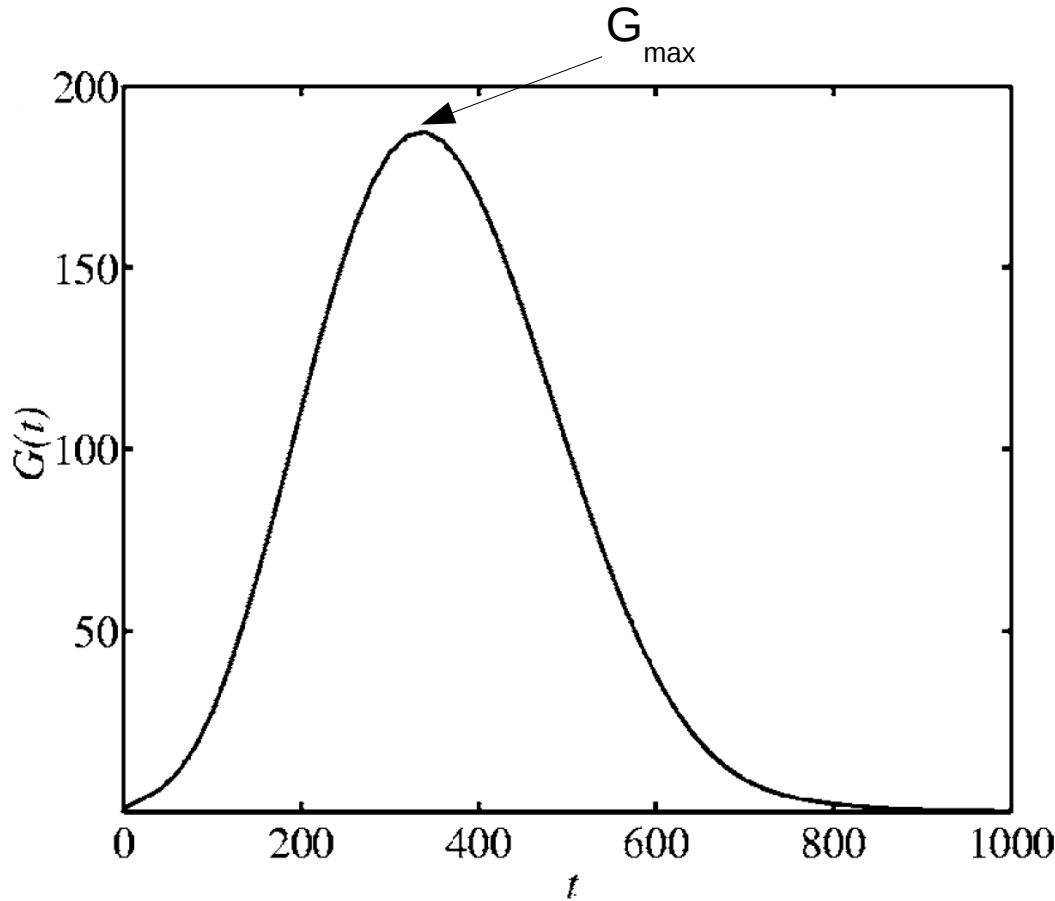


Figure: Variation of energy amplification factor for stratified viscosity and conductivity flow at $M = 2$, $Re = 2 \times 10^5$, $\alpha = 0.1$, $\beta = 0.1$.

$$\frac{\partial \tilde{\mathbf{q}}}{\partial t} = -i\mathcal{L}\tilde{\mathbf{q}}$$

$$\hat{\mathbf{q}}(x, y, z, t) = \tilde{\mathbf{q}}(y, t) \exp[i(\alpha x + \beta z)]$$

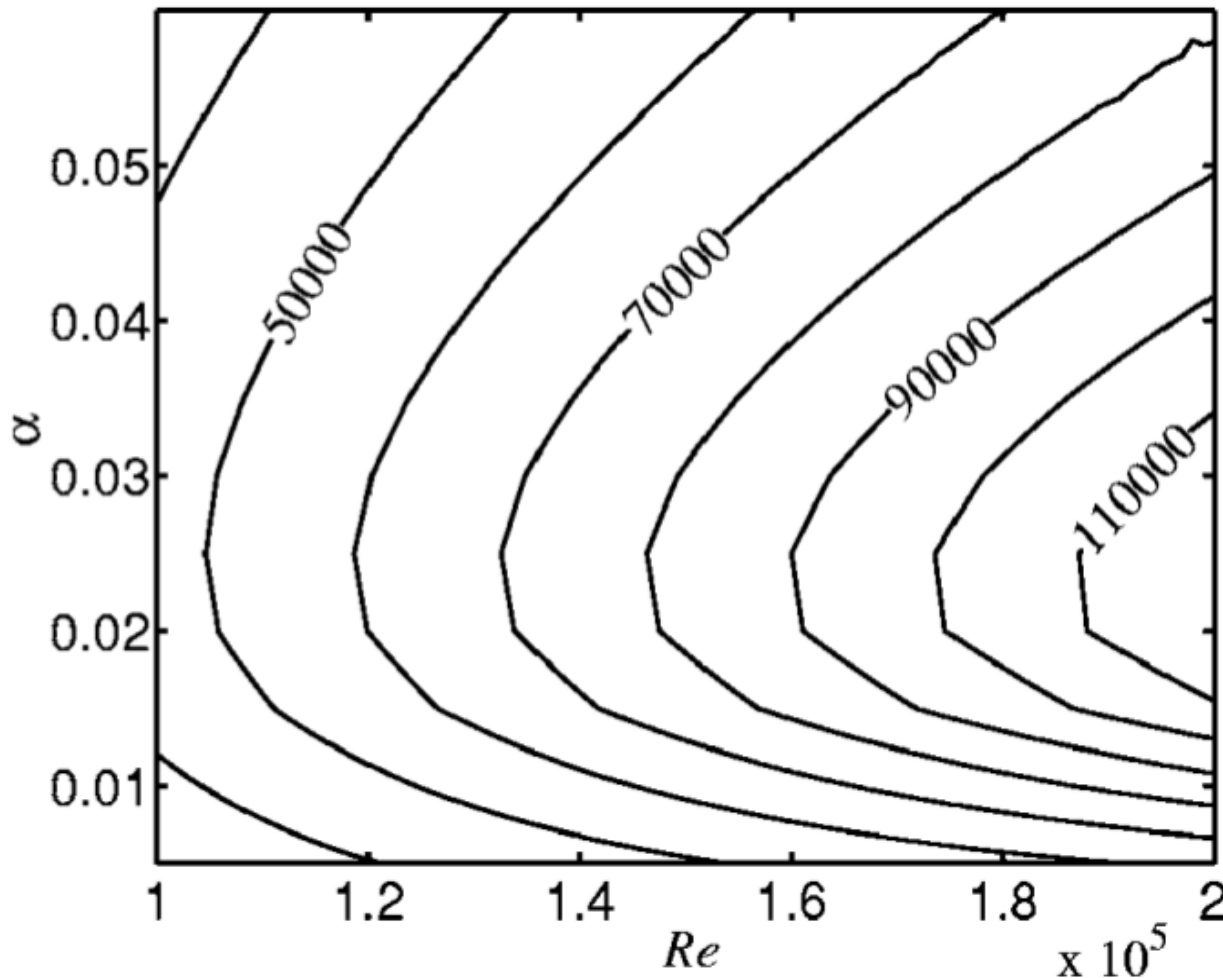
$$\mathcal{E}(\alpha, \beta, t) = \int_0^1 \tilde{\mathbf{q}}^\dagger(y, t) \mathcal{M} \tilde{\mathbf{q}}(y, t) dy$$

$$\mathcal{M} = \text{diag}\{\rho_0, \rho_0, \rho_0, T_0/\rho_0\gamma M^2, \rho_0/\gamma(\gamma-1)T_0M^2\}$$

$$G(t, \alpha, \beta; Re, M) \equiv G(t) = \max_{\tilde{\mathbf{q}}(0)} \frac{\mathcal{E}(\alpha, \beta, t)}{\mathcal{E}(\alpha, \beta, 0)}$$

$$G_{\max}(\alpha, \beta; Re, M) = \max_{t \geq 0} G(t, \alpha, \beta; Re, M)$$

Energy amplification factor



- $G_{\max}$ varies nonmonotonically with α for fixed Re .
- $G_{\max}$ is maximum for nonzero values of α .
- $G_{\max}$ increases with Re for any α .

Figure: Contours of maximum transient growth, $G_{\max}$, in the Re - α plane for $M = 3$ and $\beta = 3$.

Energy amplification factor

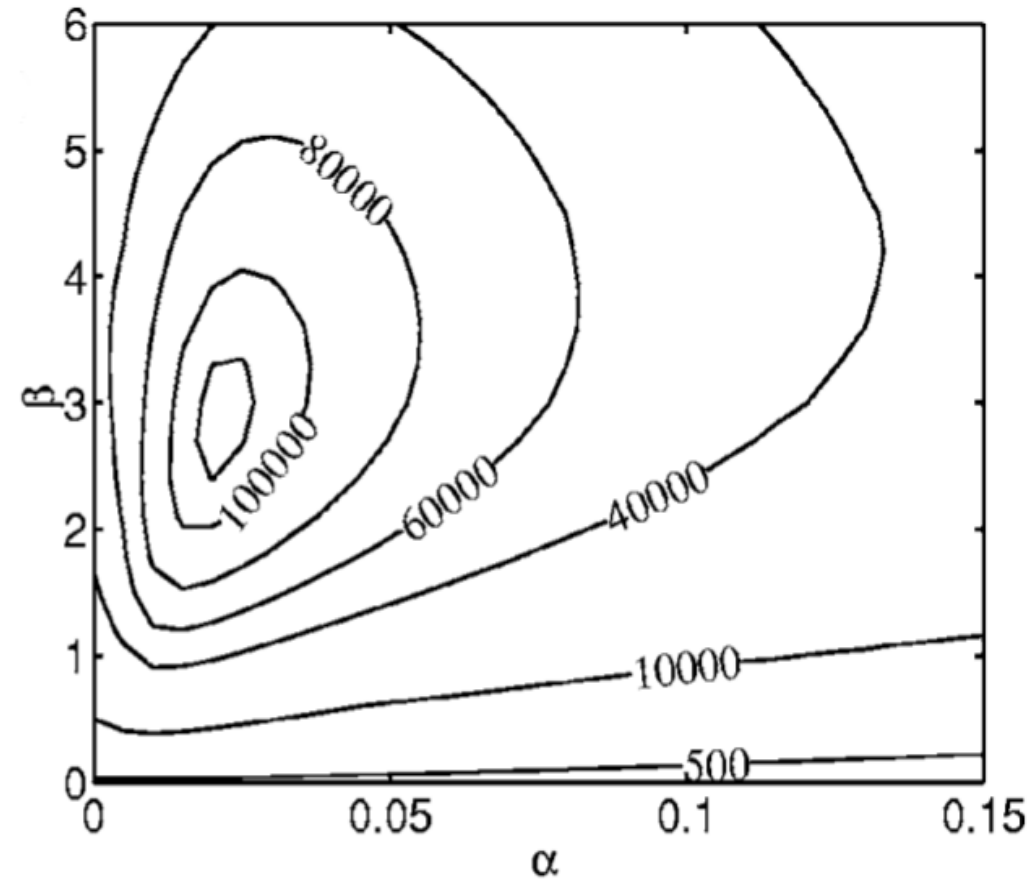


Figure: Contours of maximum transient growth, $G_{\max}$, in the α - β plane for $Re = 10^5$ and $M = 2$.

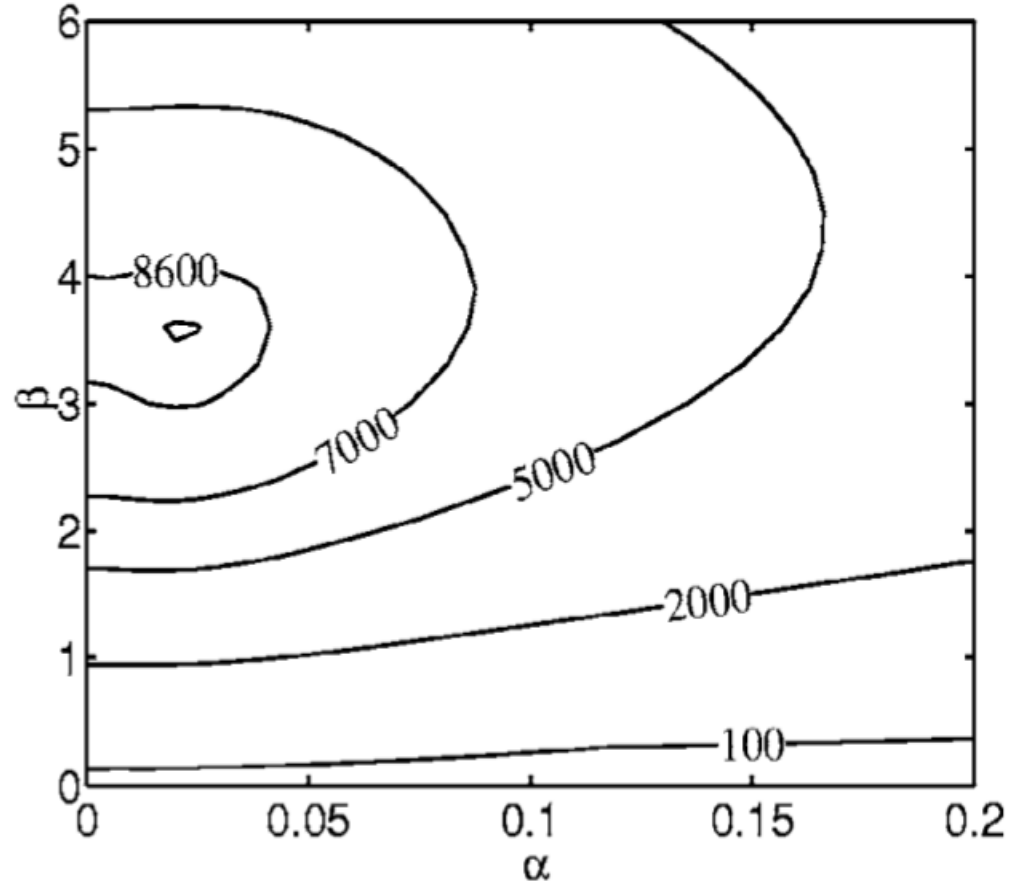


Figure: Contours of maximum transient growth, $G_{\max}$, in the α - β plane for $Re = 10^5$ and $M = 5$.

$G_{\max}$ decreases as Mach number increases from 2 to 5 for any combination of α and β .

$$G_{\text{opt}}(Re, M) = \sup_{\alpha, \beta} G_{\max}(\alpha, \beta; Re, M)$$

Optimal conditions

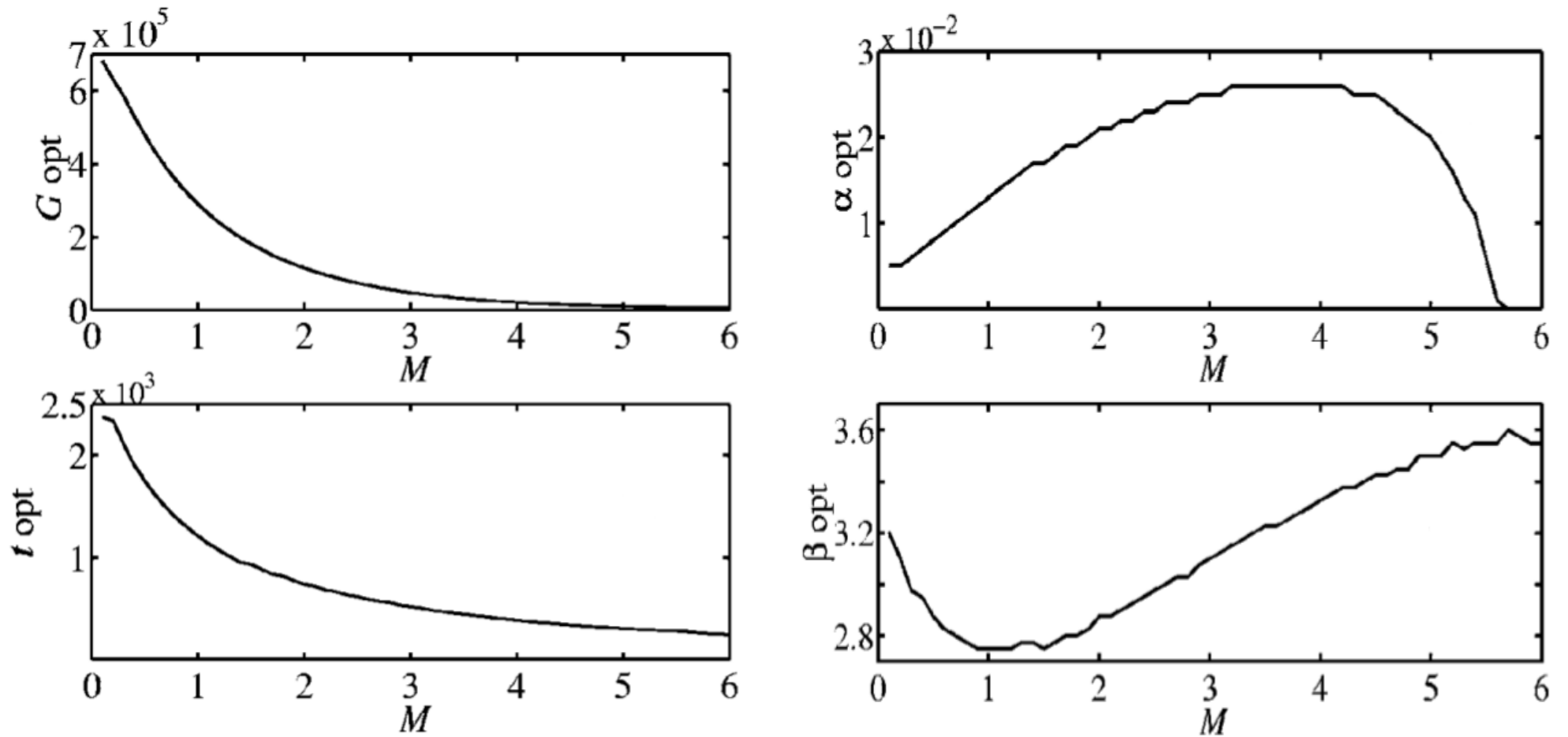
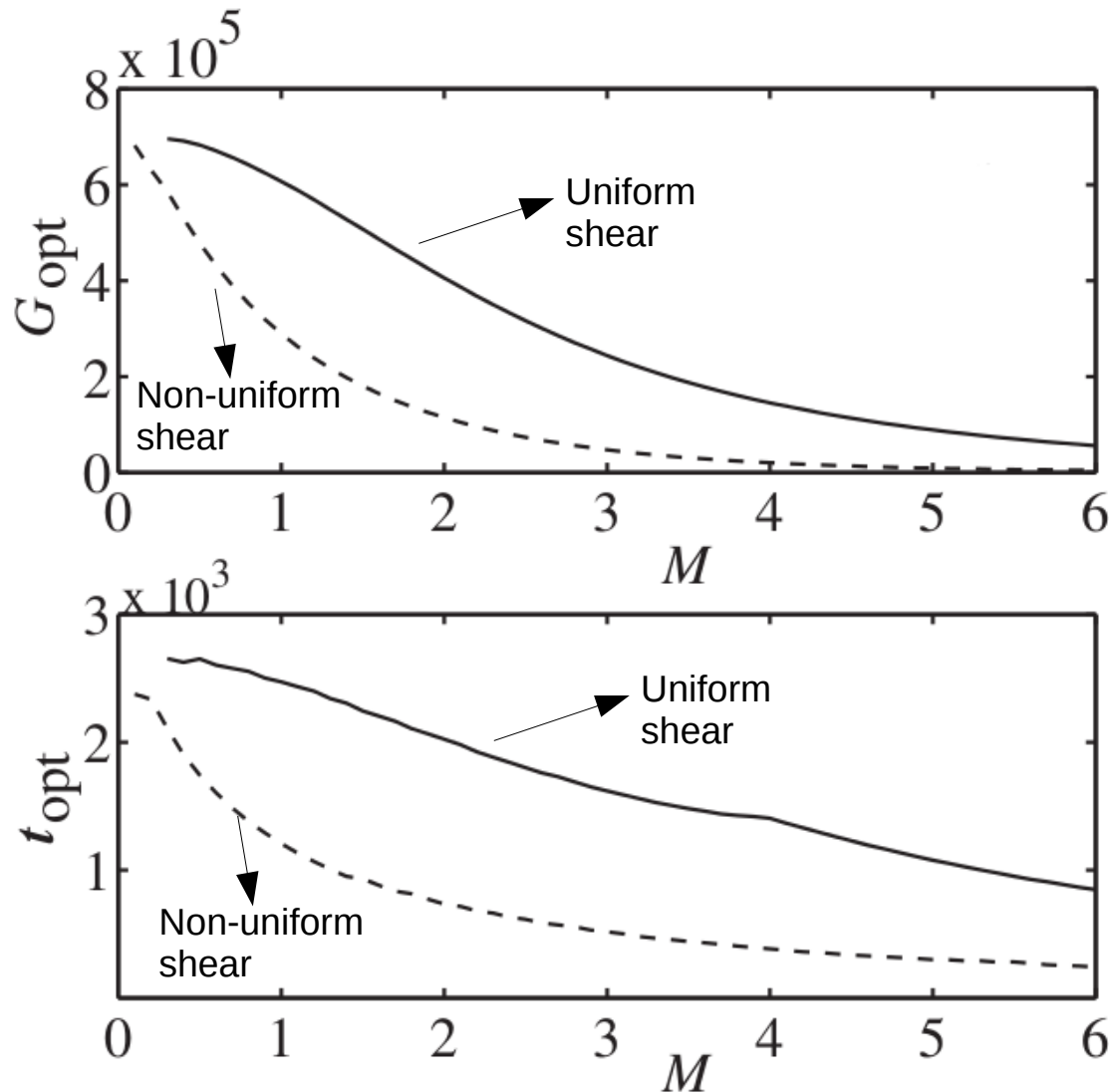


Figure: Variation of optimal energy growth, optimal time, optimal streamwise and spanwise wave number with Mach number for $\text{Re}=10^5$.

Optimal conditions



- G_{opt} is much larger for the uniform shear flow
- t_{opt} is larger by a factor of 2 or more for uniform shear flow

Figure: Variation of optimal energy growth and optimal time with Mach number for $\text{Re}=10^5$ for a flow with uniform shear and non-uniform shear.

Non-modal analysis: Summary

- Uniform shear flow is more susceptible to subcritical transitions than its nonuniform counterpart
- Viscosity stratification along with nonuniform shear would lead to a delayed subcritical transition

Thank you